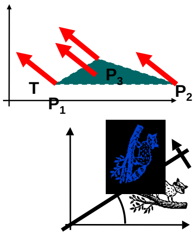
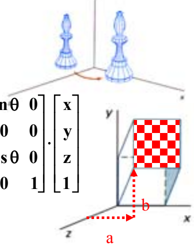
Geometric Transformations

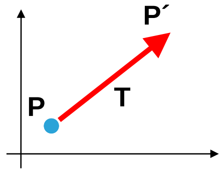


$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Basic Transformations: Translation

- translating a point from position P to position P' with translation vector T



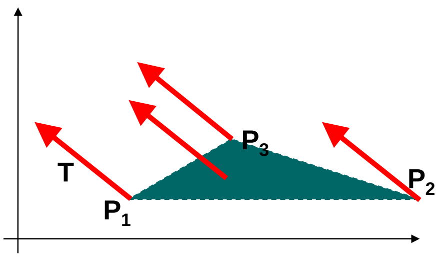
$$x' = x + t_x \qquad y' = y + t_y$$

$$\mathbf{P' = P + T}$$

notation: $P = \begin{pmatrix} x \\ y \end{pmatrix}, \; P' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \; T = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$

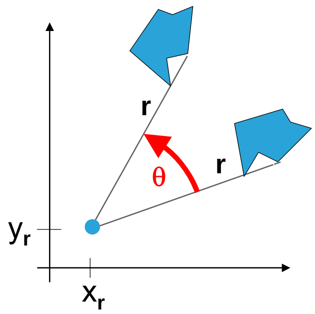
Basic Transformations: Translation

- rigid body transformation
  - object transformed by transforming boundary points



Basic Transformations: Rotation

- rotation of an object through angle θ about the pivot point (x_r, y_r)



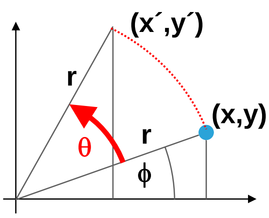
Basic Transformations: Rotation

- positive angle ⇒ ccw rotation

$x = r\cdot\cos\phi \qquad y = r\cdot\sin\phi$

$x' = r\cdot\cos(\phi + \theta)$
$\;\; = \underline{r\cdot\cos\phi}\cdot\cos\theta - \underline{r\cdot\sin\phi}\cdot\sin\theta$
$\;\; = \;\; x \;\; \cdot\cos\theta - \;\; y \;\; \cdot\sin\theta$

$y' = r\cdot\sin(\phi + \theta)$
$\;\; = r\cdot\cos\phi\cdot\sin\theta + r\cdot\sin\phi\cdot\cos\theta$



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

Basic Transformations: Rotation

- formulation with a transformation matrix

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$P' = R \cdot P \quad \text{with} \quad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$R \cdot P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

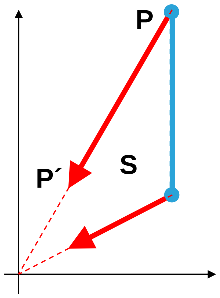
Basic Transformations: Scaling

$x' = x \cdot s_x, \quad y' = y \cdot s_y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

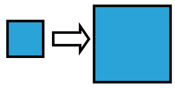
$P' = S \cdot P$

example: a line scaled using $s_x = s_y = 0.33$ is reduced in size and moved closer to the coordinate origin

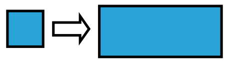


Basic Transformations: Scaling

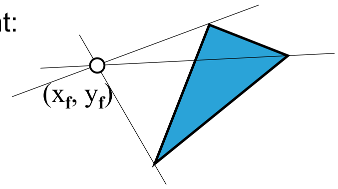
- uniform scaling: $s_x = s_y$



- differential scaling: $s_x \neq s_y$



- fixed point:



Transformation Matrices

- scaling $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$
- rotation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$
- x-mirroring $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$
- translation $(x'\ y') = (x+dx,\ y+dy)$ ... ?

Homogeneous Coordinates (1)

instead of $\begin{pmatrix} x \\ y \end{pmatrix}$ use $\begin{pmatrix} x_h \\ y_h \\ h \end{pmatrix}$ with $x = x_h/h,\ y = y_h/h$

very often $h=1$, i.e. $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

in this way all transformations can be formulated in matrix form

Homogeneous Coordinates (2)

- translation $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ $\quad P' = T(t_x, t_y) \cdot P$
- rotation $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ $\quad P' = R(\theta) \cdot P$
- scaling $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ $\quad P' = S(s_x, s_y) \cdot P$

Inverse Matrices

- translation $T^{-1}(t_x, t_y) = T(-t_x, -t_y)$
- rotation $R^{-1}(\theta) = R(-\theta)$
- scaling $S^{-1}(s_x, s_y) = S(1/s_x, 1/s_y)$

Composite Transformations (1)

n transformations are applied after each other on a point P, these transformations are represented by matrices $M_1, M_2, ..., M_n$.

$P' = M_1 \cdot P$

$P'' = M_2 \cdot P'$

...

$P^{(n)} = M_n \cdot P^{(n-1)}$

shorter: $P^{(n)} = (M_n \cdot ...(M_2 \cdot (M_1 \cdot P))...)$

Composite Transformations (2)

$P^{(n)} = (M_n \cdot ... (M_2 \cdot (M_1 \cdot P))...)$

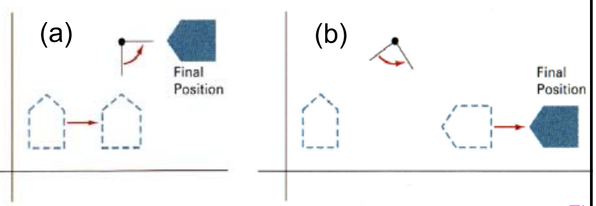
matrix multiplications are **associative**:

$$(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3)$$

(but not commutative: $M_1 \cdot M_2 \neq M_2 \cdot M_1$)

Transformations are not commutative!

Reversing the order in which a sequence of transformations is performed may affect the transformed position of an object.
In (a), an object is first translated, then rotated.
In (b), the object is rotated first, then translated.



Composite Transformations (2)

$P^{(n)} = (M_n \cdot ... (M_2 \cdot (M_1 \cdot P))...)$

matrix multiplications are **associative**:

$$(M_1 \cdot M_2) \cdot M_3 = M_1 \cdot (M_2 \cdot M_3)$$

(but not commutative: $M_1 \cdot M_2 \neq M_2 \cdot M_1$)

therefore the total transformation can also be written as: $P^{(n)} = (M_n \cdot ... \cdot M_2 \cdot M_1) \cdot P$

constant for whole images, objects, etc.!!!

Composite Transformations (3)

simple composite transformations

- composite translations

$$T(t_{x2},t_{y2}) \cdot T(t_{x1},t_{y1}) = T(t_{x1}+t_{x2}, t_{y1}+t_{y2})$$

- composite rotations

$$R(\theta_2) \cdot R(\theta_1) = R(\theta_1 + \theta_2)$$

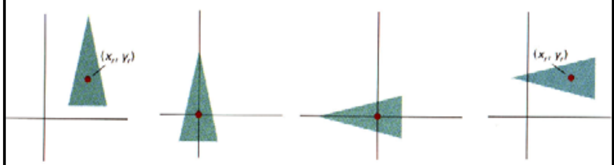
- composite scaling

$$S(s_{x2},s_{y2}) \cdot S(s_{x1},s_{y1}) = S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2})$$

Composite Transformations (4)

- general pivot-point rotation

$$T(x_r, y_r) \cdot R(\theta) \cdot T(-x_r, -y_r) = R(x_r, y_r, \theta)$$

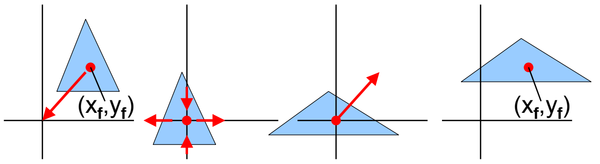


original position and pivot point | **translation of object so that pivot point is at origin** | **rotation about origin** | **translation so that the pivot point is returned**

Composite Transformations (5)

- general fixed-point scaling

$$T(x_f, y_f) \cdot S(s_x, s_y) \cdot T(-x_f, -y_f) = S(x_f, y_f, s_x, s_y)$$

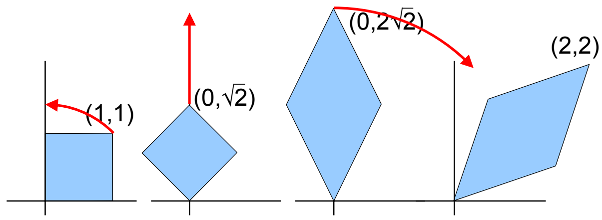


original position and fixed point	translate object so that fixed point is at origin	scale object with respect to origin	translate so that the fixed point is returned

Composite Transformations (6)

- general scaling directions

$$R^{-1}(\theta) \cdot S(s_1, s_2) \cdot R(\theta)$$



original position	after 45° rotation	after (1,2) scaling	rotate back with - 45 °

Example

translate by (3,4), then rotate by 45° and then scale up by factor 2 in x-direction

1. $M_1 = T(3,4) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

2. $M_2 = R(45°) = \begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $M_3 = S(2,1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$M = M_3 \cdot M_2 \cdot M_1$$

Example

translate by (3,4), then rotate by 45° and then scale up by factor 2 in x-direction

$$M = M_3 \cdot M_2 \cdot M_1 =$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} =$$

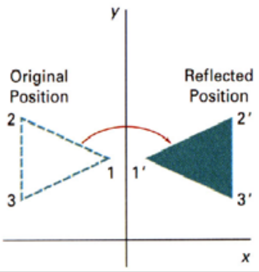
$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 45 & -\sin 45 & 3\cos 45 - 4\sin 45 \\ \sin 45 & \cos 45 & 3\sin 45 + 4\cos 45 \\ 0 & 0 & 1 \end{pmatrix} =$$

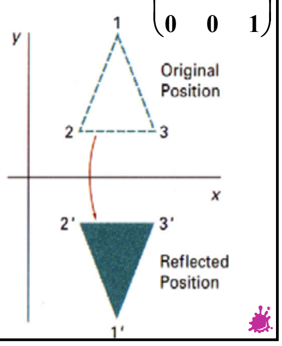
$$= \begin{pmatrix} 2\cos 45 & -2\sin 45 & 6\cos 45 - 8\sin 45 \\ \sin 45 & \cos 45 & 3\sin 45 + 4\cos 45 \\ 0 & 0 & 1 \end{pmatrix}$$

Reflection

about y-axis: $Rf_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

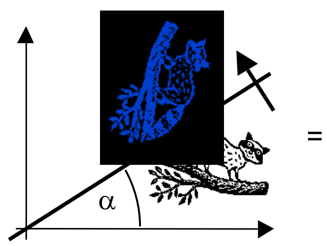
about x-axis: $Rf_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$





Example

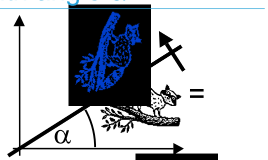
reflection about the axis with angle α

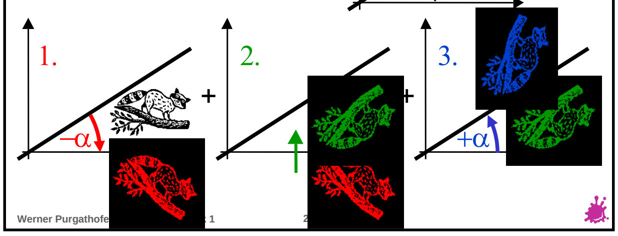


Example

reflection about the axis with angle α

1. rotation by $-\alpha$
2. mirroring about x-axis
3. rotation by $+\alpha$





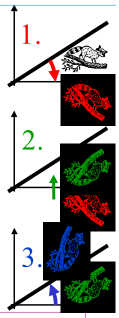
Example

reflection about the axis with angle α

1. $M_1 = R(-\alpha) = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. $M_2 = S(1,-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. $M_3 = R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$$P' = M_3 \cdot (M_2 \cdot (M_1 \cdot P)) = (M_3 \cdot M_2 \cdot M_1) \cdot P$$

Example

reflection about the axis with angle α

$M_3 \cdot M_2 \cdot M_1 =$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha & 0 \\ 2\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Computational Efficiency

- general two-dimensional transformation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} rs_{xx} & rs_{xy} & trs_x \\ rs_{yx} & rs_{yy} & trs_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$x' = x \cdot rs_{xx} + y \cdot rs_{xy} + trs_x$$

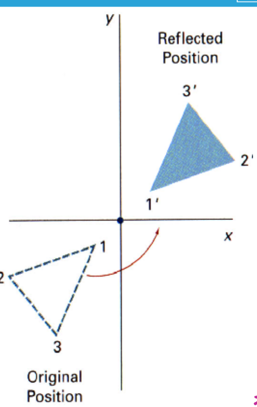
$$y' = x \cdot rs_{yx} + y \cdot rs_{yy} + trs_y$$

Other Transf.: Reflection about a point

reflection about origin

Rf_O (=R(180°)) =

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Reflection with respect to a general line

reflection with respect to the line **y=mx+b**

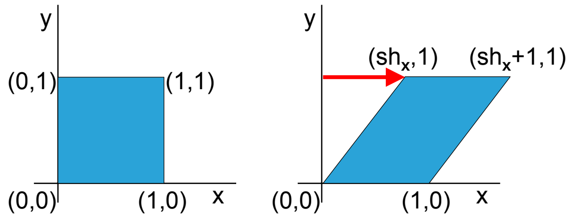
$$T(0,b) \cdot R(\theta) \cdot S(1,-1) \cdot R(-\theta) \cdot T(0,-b)$$

$$m = \tan(\theta)$$

Other Transformations: Shear (1)

- x-direction shear
  - along x-axis
  - reference line y=0

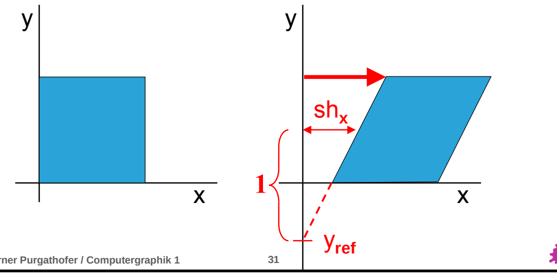
$$\begin{pmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Other Transformations: Shear (2)

- general x-direction shear
  - along x-axis
  - reference line $y=y_{ref}$

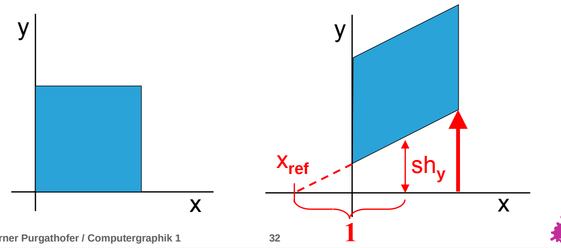
$$\begin{pmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



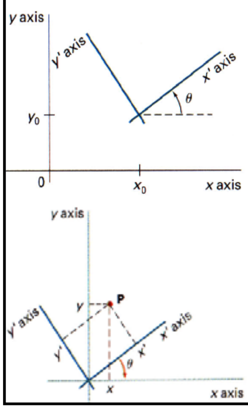
Other Transformations: Shear (3)

- general y-direction shear
  - along y-axis
  - reference line $x=x_{ref}$

$$\begin{pmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{pmatrix}$$



Transf. between Coordinate Systems



$$M_{xy,x'y'} = R(-\theta) \cdot T(-x_0, -y_0)$$

A Cartesian x'y' system positioned at (x_0,y_0) with orientation θ in an xy Cartesian system

Position of the reference frames after translating the origin of the x'y' system to the coordinate origin of the xy system

Affine Transformations

$$x' = a_{xx} x + a_{xy} y + b_x$$

$$y' = a_{yx} x + a_{yy} y + b_y$$

- collinear ⇒ points on a line stay on a line
- parallel lines ⇒ parallel lines
- ratios of distances along a line are preserved
- finite points ⇒ finite points
- any affine transformation is combination of translation, rotation, scaling, (reflection, shear)
- translation, rotation, reflection only:
  - angle, length preserving

3D Transformations

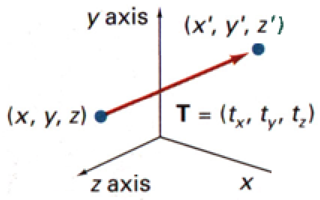
- all concepts can be extended to 3D in a straight forward way
- + projections 3D → 2D (chapter 7)

3D Translation (1)

- translation vector (t_x, t_y, t_z)

$$x' = x + t_x, \quad y' = y + t_y, \quad z' = z + t_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = T(t_x, t_y, t_z) \cdot P$$

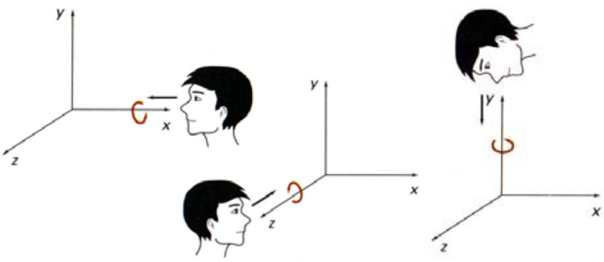
3D Translation (2)

- objects translated by translating boundary points
- inverse:

$$T^{-1}(t_x, t_y, t_z) = T(-t_x, -t_y, -t_z)$$

3D Rotation: Angle Orientation

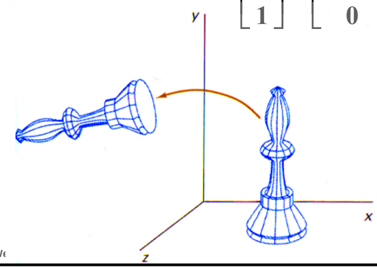
- rotation axis
- positive angle ⇒ counterclockwise rotation



3D Rotation: Coordinate Axes (z-axis)

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

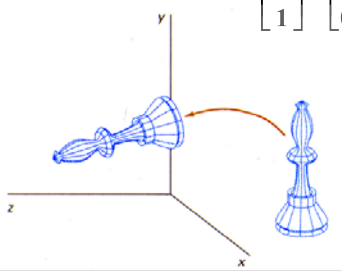
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = R_z(\theta) \cdot P$$

3D Rotation: Coordinate Axes (x-axis)

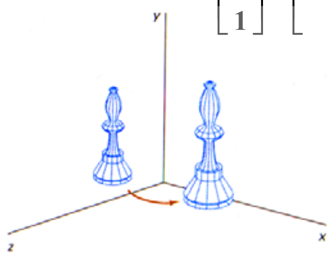
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = R_x(\theta) \cdot P$$

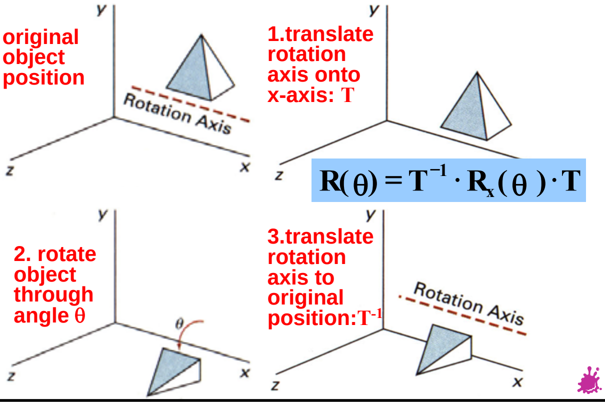
3D Rotation: Coordinate Axes (y-axis)

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = R_y(\theta) \cdot P$$

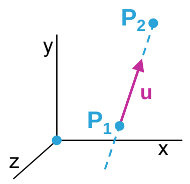
3D Rotation: Axis Parallel to x-Axis



$$\mathbf{R}(\theta) = \mathbf{T}^{-1} \cdot \mathbf{R}_x(\theta) \cdot \mathbf{T}$$

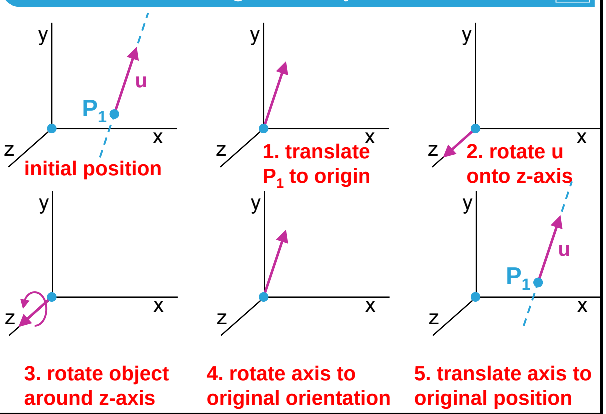
3D Rotation along Arbitrary Axis

an axis of rotation (dashed line) defined with points P_1 and P_2. The direction of the unit axis vector u determines the rotation direction.



$$u = \frac{P_2 - P_1}{|P_2 - P_1|} = (a, b, c)$$

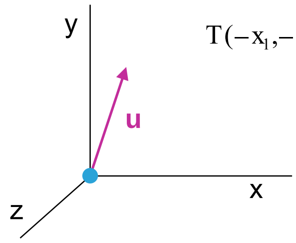
3D Rotation along Arbitrary Axis

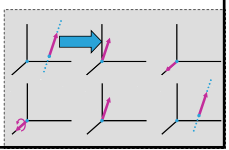


3D Rotation along Arbitrary Axis

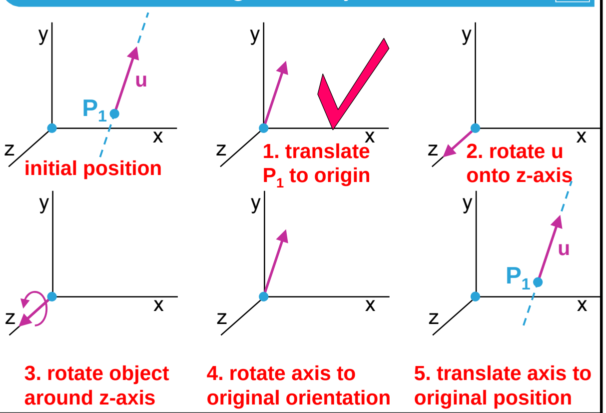
- step 1: translation $T(-x_1,-y_1,-z_1)$

$$T(-x_1,-y_1,-z_1) = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



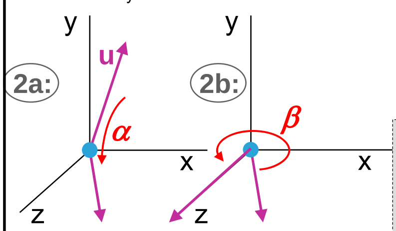


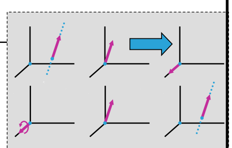
3D Rotation along Arbitrary Axis



3D Rotation along Arbitrary Axis

- step 2: rotation so that u coincides with z-axis (done with two rotations)
  - $R_x(\alpha)$: u → xz-plane
  - $R_y(\beta)$: u → z-axis





3D Rotation along Arbitrary Axis

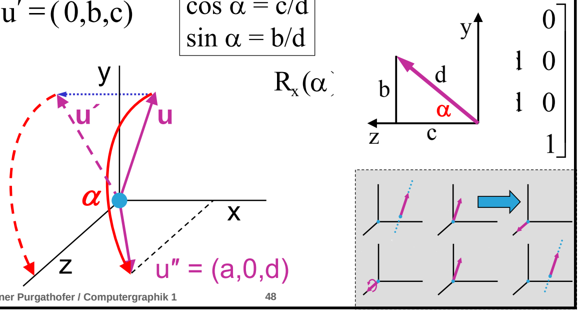
- step 2a:

$u=(a,b,c)$

$u'=(0,b,c)$

$|u'|=d=\sqrt{b^2+c^2}$

$\cos\alpha = c/d$
$\sin\alpha = b/d$



3D Rotation along Arbitrary Axis

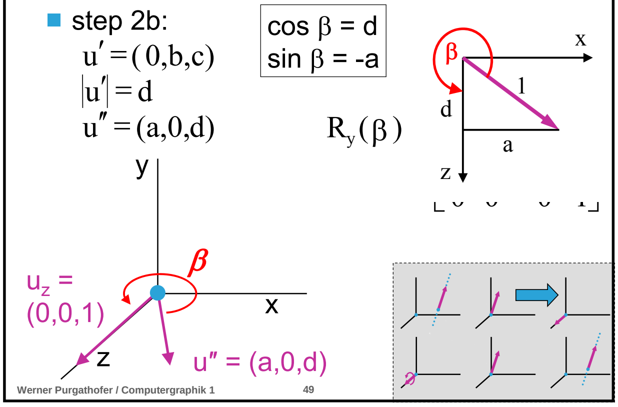
- step 2b:

$u'=(0,b,c)$

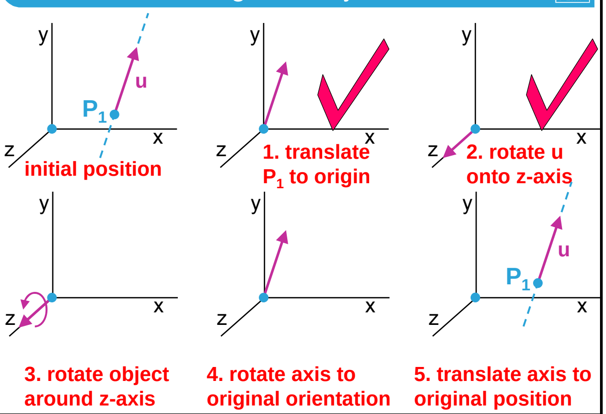
$|u'|=d$

$u''=(a,0,d)$

$\cos\beta = d$
$\sin\beta = -a$



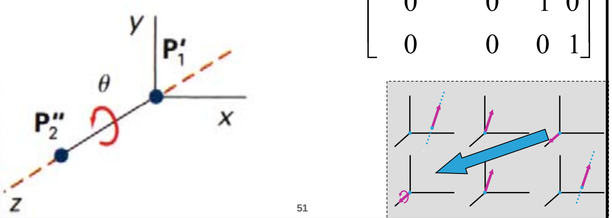
3D Rotation along Arbitrary Axis



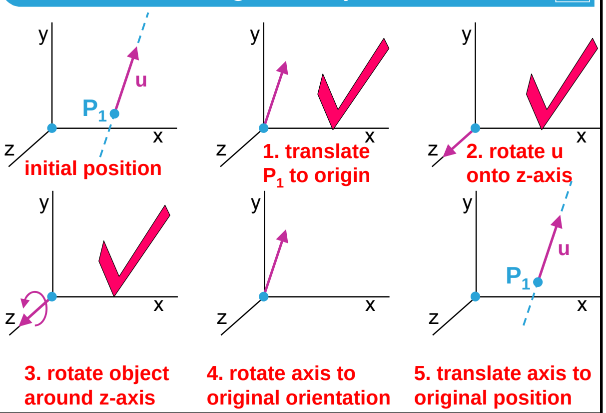
3D Rotation along Arbitrary Axis

- step 3:
  - u aligned with z-axis
  - rotation around z-axis

$$R_z(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



3D Rotation along Arbitrary Axis



3D Rotation along Arbitrary Axis

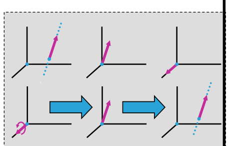
- step 4: undo rotations of step 2
- step 5: undo translation of step 1

$$R(\theta)=T^{-1}(-P_1)\cdot R_x^{-1}(\alpha)\cdot R_y^{-1}(\beta)\cdot \mathbf{R_z(\theta)}\cdot R_y(\beta)\cdot R_x(\alpha)\cdot T(-P_1)$$

steps: 5 4a 4b 3 2b 2a 1

- inverse of rotation:

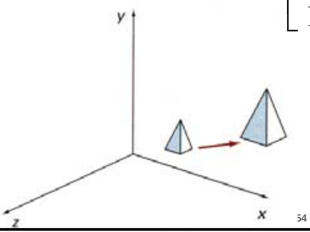
$R_x^{-1}(\theta)=R_x(-\theta)=R_x^T(\theta)$



3D Scaling with respect to Origin

doubling the size of an object also moves the object farther from the origin

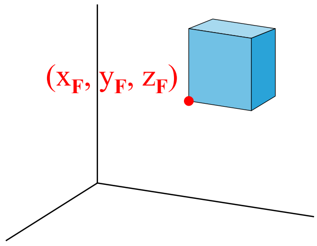
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



$$P' = S \cdot P$$

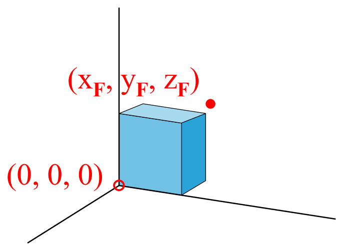
3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$



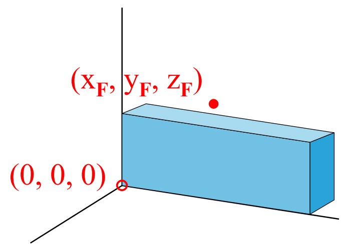
3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$



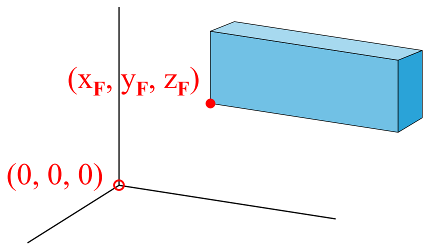
3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$

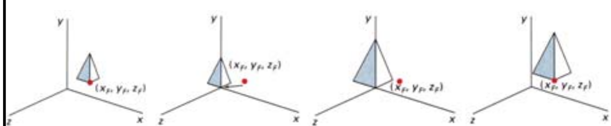


3D Scaling with other Fixed Point

$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$



3D Scaling with other Fixed Point

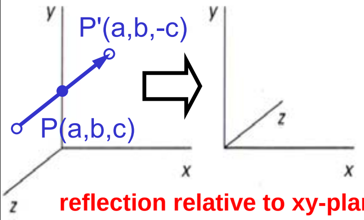


$$T(x_F, y_F, z_F) \cdot S(s_x, s_y, s_z) \cdot T(-x_F, -y_F, -z_F)$$

$$\begin{bmatrix} s_x & 0 & 0 & (1-s_x)x_F \\ 0 & s_y & 0 & (1-s_y)y_F \\ 0 & 0 & s_z & (1-s_z)z_F \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Reflection

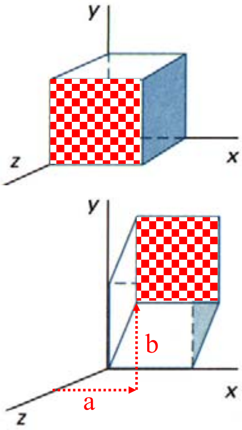
- reflection with respect to
  - point
  - line (180° rotation)
  - plane, e.g., xy-plane: RF_z



$$RF_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

reflection relative to xy-plane

3D Shear



example: shear relative to z-axis with $a=b=1$

$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$